Now let's take a slightly different approach to verifying identities. Instead of using the definitions of the trig functions, we'll use identities that we've already established to verify new ones.

Here's a simple example: Verify the identity $\cos\theta\tan^2\theta + \cos\theta = \sec\theta$.

$\cos\theta\tan^2\theta + \cos\theta = \cos\theta(\tan^2\theta + 1)$	Factor out $\cos\theta$.
$= \cos\theta\sec^2\theta$	Use the identity $\tan^2\theta + 1 = \sec^2\theta$.
$= \frac{1}{\sec\theta}\sec^2\theta$	Use the identity $\cos\theta = \frac{1}{\sec\theta}$.
$= \sec\theta$	Cancel one factor of $\sec\theta$.

Note the form of the example. We start with the more complicated side of the identity, $\cos\theta\tan^2\theta + \cos\theta$, and end with the simpler one, $\sec\theta$. At each step, we use either an algebraic technique, such as factoring or canceling, or a substitution based upon a previously established identity.

Remember, "All things through Christ."

Complete the steps to verify the identity $\sin\theta + \sin\theta\cot^2\theta = \csc\theta$.

1. $\sin\theta + \sin\theta\cot^2\theta = \sin\theta(\_\_\_\_\_\_\_\_)$ — Factor out $\sin\theta$.

$= \sin\theta\ \_\_\_\_\_\_\_\_$ — Use the identity $1 + \cot^2\theta = \csc^2\theta$.

$= \frac{\_\_\_\_}{\_\_\_\_}\csc^2\theta$ — Use the identity $\sin\theta = \frac{1}{\csc\theta}$.

$= \csc\theta$ — Cancel one factor of $\csc\theta$.

Let's try another one! In this one, you will need to use a different form of an identity that you already know. The identity $\sin^2\theta + \cos^2\theta = 1$ can be rewritten as $\sin^2\theta = 1 - \cos^2\theta$ by subtracting $\cos^2\theta$ from each side.

Complete the steps to verify the identity $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$.

2. $\cos\theta(\sec\theta - \cos\theta) = \cos\theta\ \_\_\_\_\_\_ - \cos^2\theta$ — Distribute the factor of $\cos\theta$.

$= \cos\theta\ \frac{\_\_\_\_}{\_\_\_\_} - \cos^2\theta$ — Use the identity $\sec\theta = \frac{1}{\cos\theta}$.

$= \_\_\_\_\_\_ - \cos^2\theta$ — Cancel one factor of $\cos\theta$.

$= \sin^2\theta$ — Use the identity $\sin^2\theta + \cos^2\theta = 1$ in the form $\sin^2\theta = 1 - \cos^2\theta$.

Write Luke 12:20 in your Trig Study Notebook.

Score this page. ☐	Correct mistakes. ☐	Rescore. ☐

For each activity, one identity is verified. Follow the pattern of the given verification to verify the other identity or identities. The identities in the box below may be used in the verifications.

1. $\sin\theta\csc\theta = 1$ $\qquad \cos\theta\sec\theta = 1$ $\qquad \tan\theta\cot\theta = 1$

$\sin\theta\csc\theta = \sin\theta\frac{1}{\sin\theta}$

$= 1$

2. $\frac{\sin\theta}{\sin(90° - \theta)} = \tan\theta$ $\qquad \frac{\cos\theta}{\cos(90° - \theta)} = \cot\theta$ $\qquad \frac{\sin\theta}{\cos(90° - \theta)} = 1$

$\frac{\sin\theta}{\sin(90° - \theta)} = \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$

3. $\sin\theta\sec\theta = \tan\theta$ $\qquad \cos\theta\csc\theta = \cot\theta$

$\sin\theta\sec\theta = \sin\theta\frac{1}{\cos\theta}$

$= \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$

4. $\sin\theta\cot\theta = \cos\theta$ $\qquad \cos\theta\tan\theta = \sin\theta$

$\sin\theta\cot\theta = \sin\theta\frac{\cos\theta}{\sin\theta}$

$= \cos\theta$

5. $\tan\theta\csc\theta = \sec\theta$ $\qquad \cot\theta\sec\theta = \csc\theta$

$\tan\theta\csc\theta = \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta}$

$= \frac{1}{\cos\theta}$

$= \sec\theta$

REMEMBER TO MEMORIZE LUKE 12:20.

Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$ $\qquad \csc\theta = \frac{1}{\sin\theta}$

$\cos\theta = \frac{1}{\sec\theta}$ $\qquad \sec\theta = \frac{1}{\cos\theta}$

$\tan\theta = \frac{1}{\cot\theta}$ $\qquad \cot\theta = \frac{1}{\tan\theta}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Quotient Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$

Cofunction Identities

$\sin(90° - \theta) = \sin(\frac{\pi}{2} - \theta) = \cos\theta$

$\cos(90° - \theta) = \cos(\frac{\pi}{2} - \theta) = \sin\theta$

Score this page. ☐	Correct mistakes. ☐	Rescore. ☐

Match each step of the verification with an explanation. Write the corresponding letter in the box.

1. $\tan\theta + \cot\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$	☐	**A** Multiply each fraction by 1.
$= \dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{\sin\theta}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\cos\theta}{\cos\theta}$	☐	**B** Use reciprocal identities.
$= \dfrac{\sin^2\theta}{\sin\theta\cos\theta} + \dfrac{\cos^2\theta}{\sin\theta\cos\theta}$	☐	**C** Add the fractions.
$= \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$	☐	**D** Use quotient identities.
$= \dfrac{1}{\sin\theta\cos\theta}$	☐	**E** Multiply each pair of fractions.
$= \dfrac{1}{\sin\theta}\cdot\dfrac{1}{\cos\theta}$	☐	**F** Rewrite the fraction as a product of two fractions.
$= \csc\theta\sec\theta$	☐	**G** Use a Pythagorean identity.

Study the verification on the left and then verify the identity on the right.

2.

$\sec\theta - \cos\theta = \sin\theta\tan\theta$	$\csc\theta - \sin\theta = \cos\theta\cot\theta$
$\sec\theta - \cos\theta = \dfrac{1}{\cos\theta} - \cos\theta$ $= \dfrac{1}{\cos\theta} - \cos\theta\cdot\dfrac{\cos\theta}{\cos\theta}$ $= \dfrac{1}{\cos\theta} - \dfrac{\cos^2\theta}{\cos\theta}$ $= \dfrac{1-\cos^2\theta}{\cos\theta}$ $= \dfrac{\sin^2\theta}{\cos\theta}$ $= \dfrac{\sin\theta}{1}\cdot\dfrac{\sin\theta}{\cos\theta}$ $= \sin\theta\tan\theta$	

Verify each identity on separate paper.

3. $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$ **4.** $\dfrac{\sin(90° - \theta)}{\cos(90° - \theta)} = \cot\theta$ **5.** $\sec\theta - \sin^2\theta\sec\theta = \cos\theta$

Score this page. ☐	Correct mistakes. ☐	Rescore. ☐

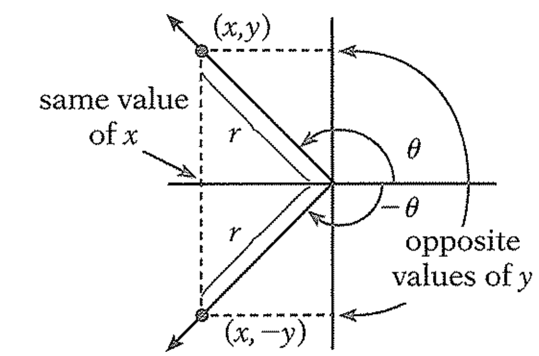
Now let's look at another set of basic identities that follow easily from the definitions of the trig functions. These identities express relationships between the value of the trig function of an angle and the value of the same trig function for the negative of the angle.

Even/Odd Identities

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$	$\cot(-\theta) = -\cot\theta$

To see how these follow from the definitions, let's consider the relationship between an angle θ and its negative. The difference between the angles θ and $-\theta$ is the direction of their rotations.

The graph at the right shows the important details about the angles. The terminal side of $-\theta$ is the reflection of the terminal side of θ across the x-axis. So, for a point (x,y) on the terminal side of θ, there is a corresponding point $(x,-y)$ on the terminal side of $-\theta$ with the same x-coordinate but opposite y-coordinate. Three of the identities are established below.

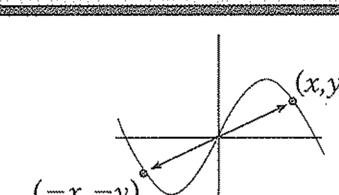


$\sin(-\theta) = \dfrac{-y}{r} = -\dfrac{y}{r} = -\sin\theta$ $\cos(-\theta) = \dfrac{x}{r} = \cos\theta$ $\tan(-\theta) = \dfrac{-y}{x} = -\dfrac{y}{x} = -\tan\theta$

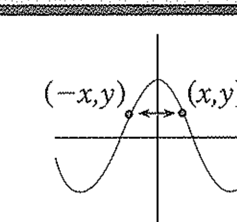
A function f satisfying $f(-x) = f(x)$ is called an ***even*** function. The identities above show us that cosine and secant are even functions. A function f satisfying $f(-x) = -f(x)$ is called an ***odd*** function. The remaining four trig functions are odd functions.

Odd functions have graphs that are symmetric with respect to the origin as shown in the graph of $y = \sin x$.

Even functions have graphs that are symmetric with respect to the y-axis as shown in the graph of $y = \cos x$.



$y = \sin x$



$y = \cos x$

Try graphing $y = \sin(-x)$ and $y = -\sin x$ to verify that you get identical graphs. Remember that this is a way to help you see that the equation $\sin(-\theta) = -\sin\theta$ is an identity.

The even/odd identities allow us to rewrite any trig expression with a negative angle measure as a trig expression with a positive angle measure.

$\sin(-40°) = -\sin 40°$ $\cos(-40°) = \cos 40°$ $\tan(-40°) = -\tan 40°$

Use an even/odd identity to rewrite each trig expression using a positive angle measure.

1. $\sin(-100°) =$ ________ **2.** $\cos(-100°) =$ ________ **3.** $\tan(-100°) =$ ________

4. $\csc(-100°) =$ ________ **5.** $\sec(-100°) =$ ________ **6.** $\cot(-100°) =$ ________

7. $\sin(-\frac{\pi}{12}) =$ ________ **8.** $\cos(-\frac{\pi}{12}) =$ ________ **9.** $\tan(-\frac{\pi}{12}) =$ ________

10. $\csc(-\frac{\pi}{12}) =$ ________ **11.** $\sec(-\frac{\pi}{12}) =$ ________ **12.** $\cot(-\frac{\pi}{12}) =$ ________

Identify each trig function as either *even* or *odd*.

13. sine ________ **14.** cosine ________ **15.** tangent ________

16. cosecant ________ **17.** secant ________ **18.** cotangent ________

Match each trig function with its graph and state whether the function is symmetric with respect to the *y-axis* or with respect to the *origin*. Use your Trig Study Notebook to help you identify the graphs if you need to.

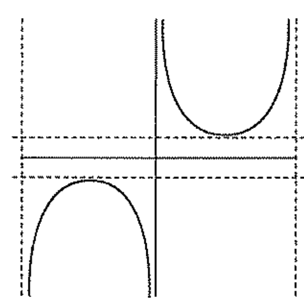
19. $y = \tan x$
Graph: ____
Symmetric with respect to the ________

20. $y = \csc x$
Graph: ____
Symmetric with respect to the ________

21. $y = \sec x$
Graph: ____
Symmetric with respect to the ________

22. $y = \cot x$
Graph: ____
Symmetric with respect to the ________

A



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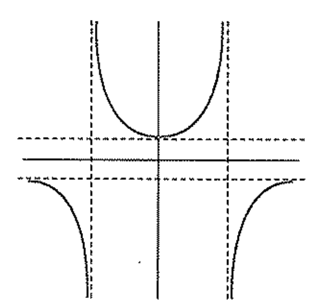

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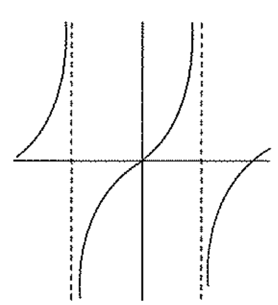

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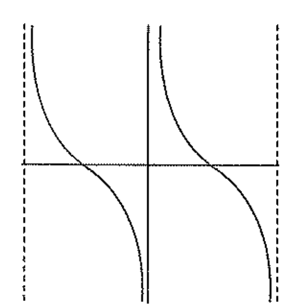


Score this page. ☐	Correct mistakes. ☐	Rescore. ☐

The next two identities that we will study are different than the ones we've seen so far in that they involve two distinct angles. These identities will allow us to express the sine or cosine of the sum of two angles in terms of the sines and cosines of the individual angles.

Sum Formulas

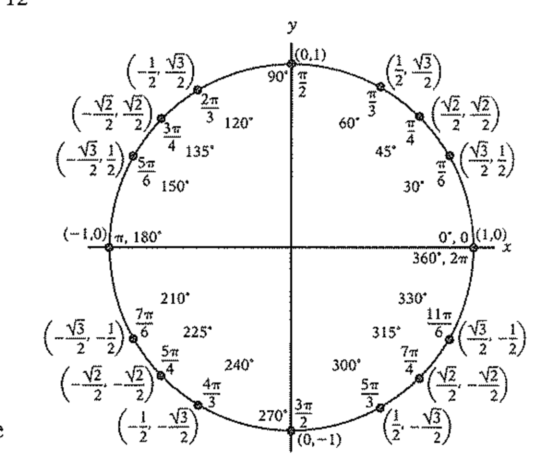
TRIG NOTES

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

These identities are more difficult to verify because they involve two angles. Therefore, in this course, we will accept the sum formulas without proof.

The sum formulas allow us to find the exact values of sine and cosine for any angle that can be expressed as a sum of our special angles. Remember that our special angles are multiples of 30°, 45°, and 60° and their equivalents in radians. The angle measures 75°, 195°, and $\frac{7\pi}{12}$ are shown below expressed as sums of special angles.

$75° = 45° + 30°$ $195° = 150° + 45°$ $\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$



Now we will find sin 75° using the sum formula for sine.

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Fill in the blanks to find sin 195° and cos $\frac{7\pi}{12}$. The unit circle is shown at the right for your convenience.

1. $\sin 195° = \sin(150° +$ ____ $) = \sin 150° \cos$ ____ $+ \cos$ ____ $\sin$ ____

$= \frac{1}{2}\cdot$ ____ $+ \left(-\frac{\sqrt{3}}{2}\right)\cdot$ ____ $=$ ____ $- \frac{\sqrt{6}}{4} =$ ________

2. $\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \right.$ ____ $\left.\right) = \cos\frac{\pi}{3}\cos$ ____ $- \sin$ ____ $\sin$ ____

$=$ ____ $\cdot\frac{\sqrt{2}}{2} -$ ____ $\cdot$ ____ $=$ ____ $-$ ____ $=$ ________

Use a sum formula to find the exact value of each expression.

3. $\cos 75° =$ ________ **4.** $\sin 165° =$ ________ **5.** $\sin\frac{13\pi}{12} =$ ________

Score this page. ☐	Correct mistakes. ☐	Rescore. ☐

The sum formulas together with the even/odd identities can be used to establish our next two identities.

Difference Formulas

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\qquad$ $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

The key to verifying each of these identities is to write $\alpha - \beta$ as $\alpha + (-\beta)$.

$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$

$= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$ — Use the sum formula for sine.

$= \sin\alpha\cos\beta + \cos\alpha(-\sin\beta)$ — Use the even/odd identities.

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Fill in the blanks in the verification of $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

1. $\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$

$= \cos\alpha$ __________ $- \sin\alpha$ __________ — Use the sum formula for cosine.

$= \cos\alpha$ __________ $- \sin\alpha$ (__________) — Use the even/odd identities.

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

The difference formulas allow us to find the exact values of the sines and cosines of angles that can be expressed as the difference of two of our special angles. For example, $15° = 45° - 30°$, so we can use the difference formulas to find the exact values of $\sin 15°$ and $\cos 15°$.

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

You understand it now. Keep going!

Fill in the blanks to find the exact value of $\cos 15°$.

2. $\cos 15° = \cos(45° -$ _______ $) = \cos 45° \cos$ _______ $+ \sin$ _______ $\sin$ _______

$= \frac{\sqrt{2}}{2}\cdot$ _______ $+ \frac{\sqrt{2}}{2}\cdot$ _______ $=$ _______ $+ \frac{\sqrt{2}}{4} =$ ______________

Use a sum or a difference formula to find the exact value of each expression.

3. $\cos 105° =$ ______________

4. $\sin 105° =$ ______________

5. $\sin 285° =$ ______________

6. $\cos 285° =$ ______________

7. $\cos\frac{17\pi}{12} =$ ______________

8. $\sin\frac{17\pi}{12} =$ ______________

Score this page. ☐	Correct mistakes. ☐	Rescore. ☐